arXiv:1101.2653v2 [math-ph] 21 Feb 2011

Geometric extensions of many-particle Hardy inequalities

Douglas Lundholm*

Department of Mathematical Sciences, University of Copenhagen
Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark

Abstract

The so-called ground state representation for the Laplacian on a subdomain of $\mathbb{R}^n$ is used to derive certain many-particle Hardy inequalities in a simple and systematic way. This includes geometric extensions of the standard Hardy inequalities to involve volumes of simplices spanned by a subset of points. Clifford/multilinear algebra is employed to simplify geometric computations.

1 Introduction

During the past century, Hardy inequalities have appeared in a variety of different forms in the literature and have played an important role in analysis and mathematical physics (see e.g. the books [10, 4] and the reviews in [1, 12]). The standard Hardy inequality associated to the Laplacian in $\mathbb{R}^d$, $d \geq 3$, is given by

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \geq C_d \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \, dx, \tag{1}$$

with the sharp constant $C_d = \frac{(d-2)^2}{4}$, and any function u in the Sobolev space $H^1(\mathbb{R}^d)$. It states explicitly that the Laplace operator on $\mathbb{R}^d$ is not only non-negative, but in a sense strictly positive, in that it is bounded below by a potential which even increases in strength unboundedly as the distance to the origin tends to zero. In quantum mechanics, this is a concrete manifestation of the uncertainty principle, and inequalities of this form have been crucial for e.g. rigorous proofs for the stability of matter (see e.g. [6]). In such many-particle contexts it becomes relevant to consider extensions of (1) involving mutual distances between a (possibly large) number, say N, of particles. Also the sharp values of the corresponding constants as well as their dependence on N are relevant for physical applications.

*e-mail: lundholm@math.ku.dk

In [3], a number of such many-particle Hardy inequalities were studied, both for non-interacting bosonic and fermionic particles (i.e. completely symmetric resp. antisymmetric wavefunctions), as well as for magnetically interacting two-dimensional particles, and the large-N optimality of the derived constants was determined in the bosonic case. In the fermionic case, the optimal large-N behaviour of the corresponding constants was determined in [2].

In this note we will focus on the bosonic case and use the so-called ground state representation for the Laplacian on a subdomain of $\mathbb{R}^n$ to derive the standard (bosonic) many-particle Hardy inequalities in a simple and systematic way. Using multilinear and Clifford algebra, the approach we take straightforwardly generalizes to other types of many-particle Hardy inequalities, involving geometric relatives and higher-dimensional analogs of distances between particles. Also the case of critical dimension, which for the standard bosonic case is equal to two, is covered, and corresponding inequalities involving logarithms found. We point out that some of the generalizations presented have been considered earlier by Laptev et. al. [5]. A more general discussion on the systematic application of the ground state representation, as well as numerous other examples of its usefulness, will be reviewed elsewhere [8].

The paper is organised as follows. In Section 2 we state the preliminary setup which allows for a straightforward and systematic derivation of the results. The main results are given in Section 3 as ground state representations for the conventional many-particle Hardy inequalities in all dimensions (Theorems 4-7), for some alternative geometric cases where the origin is singled out as a special point (Theorems 8-10), and finally for inequalities involving volumes of simplices of points (Theorems 11-13). Some computations involving multilinear algebra, and a brief note on the sharpness of the derived constants, have been placed in an appendix.

Acknowledgements. I am thankful to Fabian Portmann for continuous feedback during the writing of this manuscript and for collaboration on closely related subjects. I would also like to thank Bergfinnur Durhuus, Ari Laptev, Enno Lenzmann, and Jan Philip Solovej for discussions. This work was supported by the Danish National Research Foundation and Nordita.

2 Preliminaries: single-particle Hardy inequalities

As a preparation, we start by recalling the ground state representation in a form which is well suited for our applications, and use it to derive the standard single-particle Hardy inequalities w.r.t. a point and a higher-dimensional subspace in $\mathbb{R}^d$.

2.1 The ground state representation

We have the following simple but general *ground state representation* (GSR) for the (Dirichlet) Laplacian on a domain in $\mathbb{R}^n$:

Proposition 1 (GSR). *Let Ω be an open set in $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}_+ := (0, \infty)$ be twice differentiable. Then, for any $u \in C_0^\infty(\Omega)$ and $\alpha \in \mathbb{R}$,*

$$\int_\Omega |\nabla u|^2\, dx = \int_\Omega \left(\alpha(1-\alpha)\frac{|\nabla f|^2}{f^2} + \alpha \frac{-\Delta f}{f} \right) |u|^2\, dx + \int_\Omega |\nabla v|^2 f^{2\alpha}\, dx, \quad (2)$$

where $v := f^{-\alpha} u$.

Proof. We have for $u = f^\alpha v$ that $\nabla u = \alpha f^{\alpha-1}(\nabla f) v + f^\alpha \nabla v$, hence

$$|\nabla u|^2 = \alpha^2 f^{2(\alpha-1)} |\nabla f|^2 |v|^2 + \alpha f^{2\alpha-1} (\nabla f) \cdot \nabla |v|^2 + f^{2\alpha} |\nabla v|^2.$$

Integrating this expression over Ω, we find that the middle term on the r.h.s. produces after partial integration

$$-\alpha \int_\Omega \nabla \cdot (f^{2\alpha-1} \nabla f) |v|^2\, dx.$$

Now, using that

$$\nabla \cdot (f^{2\alpha-1} \nabla f) = (2\alpha - 1) f^{2\alpha-2} |\nabla f|^2 + f^{2\alpha-1} \Delta f,$$

and collecting the terms we arrive at (2). □

It will in the sequel be very convenient to introduce some terminology related to the ground state representation. We refer to f as the (exact or approximate) *ground state* and to α as the *GSR weight*, while the potential term in (2), i.e.

$$\alpha(1-\alpha)\frac{|\nabla f|^2}{f^2} + \alpha \frac{-\Delta f}{f}, \quad (3)$$

will be called the *GSR potential.* Note that the choice $\alpha = \frac{1}{2}$ maximizes the first term in the GSR potential, which would be the relevant term if $\Delta f = 0$ on Ω, i.e. if f is a generalized zero eigenfunction for the Laplacian on Ω. In this case the resulting GSR (2) will usually be called a Hardy GSR. We emphasize (see also [8]) that an important advantage of having the ground state representation of a Hardy inequality is that the integral term involving v in (2) provides a guide for proving sharpness (cp. Appendix B), and also opens up for improvements of the inequality.

2.2 The standard Hardy inequalities in $\mathbb{R}^d$

The obvious ground states f for the Laplacian in $\mathbb{R}^d$ are the *fundamental solutions*,

$$\begin{aligned} f_{d\neq 2}(x) &:= |x|^{-(d-2)}, &\qquad \Delta_{\mathbb{R}^d} f_d &= c_d \delta_0, \\ f_2(x) &:= \ln|x|, &\qquad \Delta_{\mathbb{R}^2} f_2 &= c_2 \delta_0, \end{aligned}$$

(where δ_0 are Dirac delta distributions supported at $\{0\}$ and c_d some irrelevant constants). Hence, for $d \neq 2$ we can consider the domain $\Omega := \mathbb{R}^d \setminus \{0\}$ on which $f := f_d > 0$ and $\Delta f = 0$. (2) is therefore optimal for $\alpha = \frac{1}{2}$, and yields the ground state representation associated to the standard Hardy inequality (1) in $\mathbb{R}^d$:

$$\int_\Omega |\nabla u|^2\, dx - \frac{(d-2)^2}{4} \int_\Omega \frac{|u|^2}{|x|^2}\, dx = \int_\Omega |\nabla v|^2 |x|^{-(d-2)}\, dx \;\geq\; 0. \qquad (4)$$

The inequality (4) holds for all $u \in C_0^\infty(\Omega)$, and hence the l.h.s. is non-negative on $H_0^1(\Omega)$ ($= H^1(\mathbb{R}^d)$ for $d \geq 2$) by closure.

For $d = 2$ we can take the domain $\Omega := \mathbb{R}^2 \setminus (\{0\} \cup \mathbb{S}^1)$ and ground state $f := |f_2|$, so that $f > 0$ and $\Delta f = 0$ on Ω. (2) then produces the corresponding two-dimensional Hardy GSR

$$\int_\Omega |\nabla u|^2\, dx - \frac{1}{4} \int_\Omega \frac{|u|^2}{|x|^2 (\ln|x|)^2}\, dx = \int_\Omega |\nabla v|^2 \big| \ln|x| \big|\, dx \;\geq\; 0. \qquad (5)$$

By closure, the l.h.s. is non-negative for $u \in H_0^1(\Omega) = H_0^1(\mathbb{R}^2 \setminus \mathbb{S}^1)$.

Remark. In the above we used that $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ is dense in $H^1(\mathbb{R}^n)$ (with the Sobolev norm) for $n \geq 2$. It is easy to see that similar density arguments are valid for $C_0^\infty(\mathbb{R}^n \setminus K)$ for the codimension $k \geq 2$ subsets K we consider in the following, typically being finite unions of closed smooth or cone-like submanifolds of dimension $n - k$.

2.3 Hardy inequalities outside subspaces

Let us also briefly recall the generalizations of the standard Hardy inequalities (4)-(5) w.r.t. the point $\{0\}$, to corresponding inequalities w.r.t. any linear subspace in $\mathbb{R}^d$. For later purposes, it is most convenient to state and derive these GSR in the language of *geometric algebra* (involving the Clifford algebra over $\mathbb{R}^d$; see [9], or Appendix A).

Let $A := \boldsymbol{a}_1 \wedge \ldots \wedge \boldsymbol{a}_p \neq 0$ be a p-blade, $0 \leq p < d$, i.e. an exterior product of p vectors $\boldsymbol{a}_j \in \mathbb{R}^d$, representing the oriented p-dimensional linear subspace $\bar{A} \subseteq \mathbb{R}^d$ spanned by $\{\boldsymbol{a}_j\}$, with magnitude $|A|$. Note that the $p+1$-blade $\boldsymbol{x} \wedge A = 0$ if and only if $\boldsymbol{x} \in \bar{A}$. Further, $\delta(\boldsymbol{x}) := |\boldsymbol{x} \wedge A||A|^{-1}$ is the minimal distance from $\boldsymbol{x}$ to $\bar{A}$, while the Clifford product $(\boldsymbol{x} \wedge A)A^{-1} = (1 - P_{\bar{A}})\boldsymbol{x}$, where $P_{\bar{A}}$ is the orthogonal projection on $\bar{A}$. We then have the following

simple Hardy GSRs, which reduce to (4) resp. (5) for $p = 0$, and which will shortly be generalized to many-particle versions:

Theorem 2 ($d - p \neq 2$)**.** *Let* $\Omega := \{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x} \wedge A| > 0\}$*. Taking the ground state* $f(\boldsymbol{x}) := |\boldsymbol{x} \wedge A|^{-(d-p-2)} \propto \delta(\boldsymbol{x})^{-(d-p-2)}$ *we obtain*

$$\int_\Omega |\nabla u|^2 \, dx - \frac{(d-p-2)^2}{4} \int_\Omega \frac{|A|^2}{|\boldsymbol{x} \wedge A|^2} |u|^2 \, dx = \int_\Omega |\nabla f^{-\frac{1}{2}} u|^2 f \, dx \ \geq \ 0, \tag{6}$$

for $u \in C_0^\infty(\Omega)$*. The corresponding Hardy inequality for the l.h.s. holds for* $u \in H_0^1(\Omega)$ *(*$= H^1(\mathbb{R}^d)$ *for* $d - p \geq 2$*).*

Proof. By choosing a basis and coordinate system appropriately, one easily computes that $\Delta \delta^{-(d-p-2)} = 0$ on Ω (f is a fundamental solution to the Laplacian w.r.t. the subspace $\bar{A}$). Furthermore, $\nabla f / f = -(d-p-2)\nabla\delta/\delta$, and it follows that the optimal weight is the standard Hardy $\alpha = \frac{1}{2}$, with $|\nabla f|^2/f^2 = (d-p-2)^2/\delta^2$.

Alternatively, by employing geometric algebra we can avoid introducing coordinates and directly obtain $\nabla f = -(d-p-2)(\boldsymbol{x} \wedge A)^{-1} A^\dagger f$ and $\Delta f = 0$ (see Appendix A). □

Theorem 3 ($d - p = 2$)**.** *Fix a length scale* $R > 0$ *and consider* $\Omega := \{\boldsymbol{x} \in \mathbb{R}^d : 0 < |\boldsymbol{x} \wedge A|/R \neq 1\}$*. Taking* $f(\boldsymbol{x}) := \left| \ln \frac{1}{R} |\boldsymbol{x} \wedge A| \right|$ *we obtain*

$$\int_\Omega |\nabla u|^2 \, dx - \frac{1}{4} \int_\Omega \frac{|A|^2}{|\boldsymbol{x} \wedge A|^2 (\ln \frac{1}{R} |\boldsymbol{x} \wedge A|)^2} |u|^2 \, dx = \int_\Omega |\nabla f^{-\frac{1}{2}} u|^2 f \, dx \ \geq \ 0, \tag{7}$$

for $u \in C_0^\infty(\Omega)$*.*

Proof. Here we have $\nabla f = \pm(\boldsymbol{x} \wedge A)^{-1} A$ (with a sign depending on p and which part of Ω we consider) and $\Delta f = \pm(d-p-2)|A|^2 |\boldsymbol{x} \wedge A|^{-2} = 0$ (see Appendix A). □

Remark. Due to the invariance of the Laplacian under translations, corresponding GSR of course hold also for affine subspaces $\boldsymbol{a} + \bar{A}$, $\boldsymbol{a} \in \mathbb{R}^d$, simply by translation $\boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{a}$ and considering $\Omega := \{|(\boldsymbol{x} - \boldsymbol{a}) \wedge A| > 0\}$ (analogously for $d - p = 2$). Furthermore, we note that the constants in (6) are sharp, just as for $p = 0$ (cp. Appendix B).

3 Many-particle Hardy inequalities

We turn now to a systematic application of the GSR with approximate ground states to the setting of many-particle Hardy inequalities.

3.1 Conventional many-particle inequalities

Consider a tuple $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ of N points, or *particles*, in $\mathbb{R}^d$. We define the distance $r_{ij} := |\boldsymbol{x}_i - \boldsymbol{x}_j|$ between two particles, and the *circumradius* R_{ijk} associated to three non-coincident particles,

$$\frac{1}{2R_{ijk}^2} := \sum_{\text{cyclic in } i,j,k} (\boldsymbol{x}_i - \boldsymbol{x}_j)^{-1} \cdot (\boldsymbol{x}_i - \boldsymbol{x}_k)^{-1},$$

i.e. the radius of the circle that the particles $\boldsymbol{x}_i$, $\boldsymbol{x}_j$, $\boldsymbol{x}_k$ inscribe ($R_{ijk} := \infty$ for collinear particles, for which the r.h.s. is zero; cp. Lemma 3.2 in [3]). Let us first consider the total separation measured by the distance-squared between all pairs of particles:

Theorem 4 (Total separation of $N \geq 2$ particles)**.** *Let*

$$\Omega := \mathbb{R}^{dN} \setminus \{\boldsymbol{x}_1 = \boldsymbol{x}_2 = \ldots = \boldsymbol{x}_N\}.$$

Taking the ground state $\rho(x)^2 := \sum_{i<j} |\boldsymbol{x}_i - \boldsymbol{x}_j|^2$ *we obtain*

$$\int_\Omega |\nabla u|^2 \, dx - N \left(\frac{N-1}{2} d - 1 \right)^2 \int_\Omega \frac{|u|^2}{\rho^2} \, dx = \int_\Omega |\nabla \rho^{-2\alpha} u|^2 \rho^{4\alpha} \, dx \geq 0, \quad (8)$$

for all $u \in C_0^\infty(\Omega)$, *with the optimal weight* $\alpha := -\frac{(N-1)d-2}{4}$.

Remark. This gives a generalization to $N \neq 3$ of (3.5) in [3]. Note that $\operatorname{codim} \Omega^c = dN - d = d(N-1)$ and hence that the corresponding Hardy inequality on $H_0^1(\Omega)$ holds on $H^1(\mathbb{R}^{dN})$ unless $d = 1$ and $N = 2$. The constant is sharp, as shown explicitly in the appendix.

Proof. One computes

$$\nabla_k \rho^2 = 2 \sum_{j \neq k} (\boldsymbol{x}_k - \boldsymbol{x}_j),$$

hence

$$\Delta \rho^2 = 2 \sum_k \sum_{j \neq k} \nabla_k \cdot (\boldsymbol{x}_k - \boldsymbol{x}_j) = 2N(N-1)d,$$

and

$$|\nabla \rho^2|^2 = 8 \sum_{i<j} r_{ij}^2 + 8 \sum_k \sum_{i<j} (\boldsymbol{x}_k - \boldsymbol{x}_i) \cdot (\boldsymbol{x}_k - \boldsymbol{x}_j) = 4N\rho^2,$$

where in the last step we used the identity

$$\sum_k \sum_{i<j} (\boldsymbol{x}_k - \boldsymbol{x}_i) \cdot (\boldsymbol{x}_k - \boldsymbol{x}_j) = \frac{N-2}{2} \sum_{i<j} |\boldsymbol{x}_i - \boldsymbol{x}_j|^2. \quad (9)$$

It follows that we have a GSR potential

$$\alpha(1-\alpha) \frac{|\nabla \rho^2|^2}{\rho^4} - \alpha \frac{\Delta \rho^2}{\rho^2} = 4N\alpha \left(\frac{2-(N-1)d}{2} - \alpha \right) \frac{1}{\rho^2},$$

which by optimization proves the theorem. $\square$

Next, we have as a special case of the following, the so-called 'standard' many-particle Hardy inequality:

Theorem 5 (Separation and circumradii of pairs and triples of particles in $d \geq 3$)**.** *Let*

$$\Omega := \{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathbb{R}^{dN} : \boldsymbol{x}_i \neq \boldsymbol{x}_j \ \forall i \neq j\}.$$

Taking the ground state $f(x) := \prod_{j<k} |\boldsymbol{x}_j - \boldsymbol{x}_k|^{-(d-2)}$ *we obtain*

$$\int_\Omega |\nabla u|^2\, dx - (d-2)^2 \int_\Omega \left(2\alpha(1-\alpha) \sum_{i<j} \frac{1}{r_{ij}^2} - \alpha^2 \sum_{i<j<k} \frac{1}{R_{ijk}^2} \right) |u|^2\, dx$$
$$= \int_\Omega |\nabla f^{-\alpha} u|^2 f^{2\alpha}\, dx \ \geq \ 0, \tag{10}$$

for all $u \in C_0^\infty(\Omega)$ *and* $\alpha \in \mathbb{R}$*. In particular, defining*

$$K_{d,N} := \sup_{x\in\Omega} \frac{\sum_{i<j<k} 1/R_{ijk}^2}{\sum_{i<j} 1/r_{ij}^2} \quad \leq N-2 < \infty, \tag{11}$$

(the upper bound following from geometric relations; cp. Lemma 3.3 in [3]), and taking the then optimal weight $\alpha := \frac{1}{2+K_{d,N}}$ *we have*

$$\int_\Omega |\nabla u|^2\, dx - \frac{(d-2)^2}{2+K_{d,N}} \sum_{i<j} \int_\Omega \frac{|u|^2}{r_{ij}^2}\, dx \geq \int_\Omega |\nabla f^{-\alpha} u|^2 f^{2\alpha}\, dx \ \geq \ 0. \tag{12}$$

On the other hand, assuming $\alpha(1-\alpha) \geq 0$ *in* (10) *and now using the bound* (11) *on the term involving* r_{ij} *we obtain*

$$\int_\Omega |\nabla u|^2\, dx - \frac{(d-2)^2}{K_{d,N}(2+K_{d,N})} \sum_{i<j<k} \int_\Omega \frac{|u|^2}{R_{ijk}^2}\, dx \geq \int_\Omega |\nabla f^{-\alpha} u|^2 f^{2\alpha}\, dx \ \geq \ 0, \tag{13}$$

again with the optimal weight $\alpha := \frac{1}{2+K_{d,N}}$.

Remark. As pointed out in [3], using the geometric relations between separation and circumradii, the $N = 3$ case of Theorem 4 also implies

$$\int_\Omega |\nabla u|^2\, dx \geq \frac{(d-1)^2}{3} \binom{N}{3}^{-1} \frac{N}{3} \sum_{i<j<k} \int_\Omega \frac{|u|^2}{R_{ijk}^2}\, dx. \tag{14}$$

Combining this with (10), one is led to maximize $\frac{\alpha(1-\alpha)}{1+c\alpha^2}$ with

$$c := \frac{3}{2} \frac{(d-2)^2}{(d-1)^2} (N-1)(N-2).$$

This results in

$$\int_\Omega |\nabla u|^2\,dx - \alpha(d-2)^2 \sum_{i<j} \int_\Omega \frac{|u|^2}{r_{ij}^2}\,dx \geq \frac{1}{1+c\alpha^2} \int_\Omega |\nabla f^{-\alpha} u|^2 f^{2\alpha}\,dx \ \geq \ 0, \tag{15}$$

with the optimal $\alpha := (1+\sqrt{1+c})^{-1}$. The Hardy inequalities corresponding to (12) and (15) were given in the form of Theorem 2.1, (4.9), (4.11) in [3].

Remark. All corresponding Hardy inequalities from (12)–(15) hold on the full space of functions $H^1(\mathbb{R}^{dN})$ since $\operatorname{codim} \Omega^c = dN - d - (N-2)d = d \geq 3$.

Remark. For $N = 2$ we simply have $\alpha = \frac{1}{2}$ and (12) and (15) reduce to (8) with the sharp constant $(d-2)^2/2$. It was also noted in [3] that the large N behaviour of the constant in (12)/(15) with $K_{d,N} \sim N$ cannot be improved.

Proof. One computes

$$\nabla_k f = -(d-2) f \sum_{j \neq k} (\boldsymbol{x}_k - \boldsymbol{x}_j)^{-1},$$

hence

$$|\nabla f|^2 = (d-2)^2 f^2 \left(2 \sum_{i<j} \frac{1}{r_{ij}^2} + \sum_{i<j<k} \frac{1}{R_{ijk}^2} \right),$$

and, due to $\Delta_{\boldsymbol{x}_k} |\boldsymbol{x}_k - \boldsymbol{x}_j|^{-(d-2)} = 0$ on Ω,

$$\Delta f = (d-2)^2 f \sum_{i<j<k} \frac{1}{R_{ijk}^2}.$$

This gives the GSR (10) in the theorem. Bounding the (in total positive) term involving R_{ijk} in that equation by the term involving r_{ij} by means of $K_{d,N}$ in (11), we obtain the total constant in (12)

$$(d-2)^2 \alpha (2(1-\alpha) - \alpha K_{d,N}) = (d-2)^2 (2 + K_{d,N})\, \alpha \left(\frac{2}{2+K_{d,N}} - \alpha \right),$$

which is optimal for $\alpha = (2 + K_{d,N})^{-1}$. (13) follows similarly. ☐

The one-dimensional case is much simpler due to collinearity:

Theorem 6 (Separation of pairs of particles in $d = 1$)**.** *Let*

$$\Omega := \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_i \neq x_j \ \forall i \neq j\}.$$

Taking the ground state $f(x) := \prod_{j<k} |x_j - x_k|$ *we obtain*

$$\int_\Omega |\nabla u|^2\,dx - \frac{1}{2} \int_\Omega \left(\sum_{i<j} \frac{1}{r_{ij}^2} \right) |u|^2\,dx = \int_\Omega |\nabla f^{-\frac{1}{2}} u|^2 \prod_{i<j} r_{ij}\,dx \ \geq \ 0, \tag{16}$$

for all $u \in C_0^\infty(\Omega)$.

Remark. This is a GSR version of Theorem 2.5 in [3]. The corresponding inequality holds for all $u \in H_0^1(\Omega)$ (note that $\operatorname{codim} \Omega^c = 1$ in this case) and is sharp.

Proof. In this case $R_{ijk} = \infty$ and $\Delta f = 0$ on Ω. Hence $\alpha = \frac{1}{2}$ optimizes the GSR. $\square$

For the two-dimensional case we fix a length scale $R > 0$ and define

$$\tilde{r}_{ij} := |\boldsymbol{x}_i - \boldsymbol{x}_j| \big| \ln \frac{1}{R} |\boldsymbol{x}_i - \boldsymbol{x}_j| \big|,$$

and $\tilde{R}_{ijk}$ by

$$\frac{1}{2\tilde{R}_{ijk}^2} := \sum_{\text{cycl}} \frac{(\boldsymbol{x}_i - \boldsymbol{x}_j)^{-1}}{\big| \ln \frac{1}{R} |\boldsymbol{x}_i - \boldsymbol{x}_j| \big|} \cdot \frac{(\boldsymbol{x}_i - \boldsymbol{x}_k)^{-1}}{\big| \ln \frac{1}{R} |\boldsymbol{x}_i - \boldsymbol{x}_k| \big|}. \tag{17}$$

Theorem 7 (Separation of pairs of particles in $d = 2$)**.** *Let*

$$\Omega := \{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathbb{R}^{2N} : \boldsymbol{x}_i \neq \boldsymbol{x}_j \ \forall i \neq j\} \cap (B_{R/2}(0))^N.$$

Taking the ground state $f(x) := \prod_{i<j} \big| \ln \frac{1}{R} |\boldsymbol{x}_i - \boldsymbol{x}_j| \big|$ *we obtain*

$$\int_\Omega |\nabla u|^2 \, dx - \int_\Omega \left(2\alpha(1-\alpha) \sum_{i<j} \frac{1}{\tilde{r}_{ij}^2} - \alpha^2 \sum_{i<j<k} \frac{1}{\tilde{R}_{ijk}^2} \right) |u|^2 \, dx$$
$$= \int_\Omega |\nabla f^{-\alpha} u|^2 f^{2\alpha} \, dx \ \geq \ 0,$$

for all $u \in C_0^\infty(\Omega)$*. Hence, if* $K_{2,N} := \sup_{x \in \Omega} \sum \tilde{R}_{ijk}^{-2} / \sum \tilde{r}_{ij}^{-2}$*, then*

$$\int_\Omega |\nabla u|^2 \, dx - \frac{1}{2 + K_{2,N}} \sum_{i<j} \int_\Omega \frac{|u|^2}{\tilde{r}_{ij}^2} \, dx \geq \int_\Omega |\nabla f^{-\alpha} u|^2 f^{2\alpha} \, dx \ \geq \ 0, \tag{18}$$

and

$$\int_\Omega |\nabla u|^2 \, dx - \frac{1}{K_{2,N}(2 + K_{2,N})} \sum_{i<j<k} \int_\Omega \frac{|u|^2}{\tilde{R}_{ijk}^2} \, dx \geq \int_\Omega |\nabla f^{-\alpha} u|^2 f^{2\alpha} \, dx \ \geq \ 0, \tag{19}$$

with $\alpha := \frac{1}{2 + K_{2,N}}$.

Remark. Here we have a rough bound $K_{2,N} \leq 2(N-2)$, which simply follows by applying the Cauchy-Schwarz inequality in $\mathbb{R}^2$ to each term in (17).

Remark. In this case we have from (18) a non-trivial two-dimensional many-particle Hardy inequality on $H_0^1(\Omega) = H_0^1(B_{R/2}(0)^N)$. This correponds to the physical situation where we consider the particles to be confined to a finite area. Also note that, despite the logarithms, this inequality gives a rough lower bound

$$\inf_{\substack{u \in H^1: \\ \|u\|_{L^2}=1}} \int_{B_{R/2}(0)^N} |\nabla u|^2 \, dx \;\geq\; \frac{\mathrm{const} \cdot \binom{N}{2}}{(2 + K_{2,N})R^2}$$

to the ground state energy of a confined gas of two-dimensional non-interacting bosonic particles, which hence is of the same form as for $d \geq 3$.

Proof. This theorem follows just as in the proof of Theorem 5, using that $\nabla_k f = f \sum_{j \neq k} \left| \ln \frac{1}{R} |\boldsymbol{x}_k - \boldsymbol{x}_j| \right|^{-1} (\boldsymbol{x}_k - \boldsymbol{x}_j)^{-1}$ and $\Delta f = f \sum_{i<j<k} \tilde{R}_{ijk}^{-2}$. □

3.2 Some other inequalities of many-particle type

Associated to the original one-dimensional Hardy inequality away from the origin is also the following 'many-particle' version:

Theorem 8. *Let $\Omega := \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_1 \ldots x_N \neq 0\}$. Taking the ground state $f(x) := \prod_{k=1}^N |x_k| \, |x|^{2(1-N)}$ we obtain*

$$\begin{aligned} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega \left(\frac{1}{4} \sum_{k=1}^N \frac{1}{x_k^2} + (N-1)^2 \frac{1}{|x|^2} \right) |u|^2 \, dx \\ = \int_\Omega |\nabla f^{-\frac{1}{2}} u|^2 \prod_{k=1}^N |x_k| \, |x|^{2(1-N)} \, dx \;\geq\; 0, \end{aligned}$$

for all $u \in C_0^\infty(\Omega)$. Hence, the l.h.s. is non-negative on $H_0^1(\Omega)$.

Remark. The corresponding Hardy inequality was proved and applied in [12].

Proof. A simple modification of Proposition 1 to involve a product $g^\alpha h^\beta$ of ground state ansätze gives the GSR potential (see also [8])

$$\alpha(1-\alpha)\frac{|\nabla g|^2}{g^2} + \alpha\frac{-\Delta g}{g} + \beta(1-\beta)\frac{|\nabla h|^2}{h^2} + \beta\frac{-\Delta h}{h} - 2\alpha\beta\frac{\nabla g \cdot \nabla h}{gh}.$$

With $g := \prod_k |x_k|$ and $h := |x|^2$, and using that $\nabla_k g = g/x_k$, $\Delta g = 0$, $\nabla_k h = 2x_k$, and $\Delta h = 2N$, we find the potential

$$\alpha(1-\alpha) \sum_k \frac{1}{x_k^2} + 4\beta \left(\frac{2 - (1+2\alpha)N}{2} - \beta \right) \frac{1}{|x|^2},$$

which with the condition that the first term be optimal yields $\alpha := \frac{1}{2}$ and $\beta := \frac{1-N}{2}$. We also note that $\operatorname{codim} \Omega^c = 1$. □

The following application of our systematic approach has some relations with *bosonic* and *supersymmetric matrix models* (see e.g. [7]). Consider an N-tuple $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ of vectors in $\mathbb{R}^d$ and define the bivectors (2-blades) $B_{ij} := \boldsymbol{x}_i \wedge \boldsymbol{x}_j$, $i < j$, with magnitudes

$$|B_{ij}| = \sqrt{|\boldsymbol{x}_i|^2|\boldsymbol{x}_j|^2 - (\boldsymbol{x}_i \cdot \boldsymbol{x}_j)^2} = |\boldsymbol{x}_i||\boldsymbol{x}_j| \sin\theta_{ij},$$

and inverses $B_{ij}^{-1} = (\boldsymbol{x}_i \wedge \boldsymbol{x}_j)^{-1} = -(\boldsymbol{x}_i \wedge \boldsymbol{x}_j)/|B_{ij}|^2$ (again, cp. [9] or Appendix A). Note that $B_{ij} = 0$ if and only if $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ are parallel. We define the corresponding geometric quantities

$$\Sigma_1(x) := \sum_{j \neq k} \left| \boldsymbol{x}_j \llcorner (\boldsymbol{x}_j \wedge \boldsymbol{x}_k)^{-1} \right|^2 = \sum_{j<k} \frac{|\boldsymbol{x}_j|^2 + |\boldsymbol{x}_k|^2}{|\boldsymbol{x}_j \wedge \boldsymbol{x}_k|^2},$$

and

$$\Sigma_2(x) := \sum_{i \neq j \neq k \neq i} \left(\boldsymbol{x}_i \llcorner (\boldsymbol{x}_i \wedge \boldsymbol{x}_k)^{-1} \right) \cdot \left(\boldsymbol{x}_j \llcorner (\boldsymbol{x}_j \wedge \boldsymbol{x}_k)^{-1} \right),$$

where we note that

$$-\boldsymbol{x}_i \llcorner (\boldsymbol{x}_i \wedge \boldsymbol{x}_k)^{-1} = \boldsymbol{x}_i (\boldsymbol{x}_i \wedge \boldsymbol{x}_k) |\boldsymbol{x}_i \wedge \boldsymbol{x}_k|^{-2}$$

is the vector in the (oriented) plane $\boldsymbol{x}_i \wedge \boldsymbol{x}_k$ obtained by rotating $\boldsymbol{x}_i$ by $90°$ towards $\boldsymbol{x}_k$ and rescaling by the inverse area $|\boldsymbol{x}_i \wedge \boldsymbol{x}_k|^{-1}$. We then have the following result, which one could think of as a higher-dimensional combination of Theorem 2 with Theorem 5, involving 1-dimensional subspaces ($A = \boldsymbol{x}_j$) instead of 0-dimensional ($A = 1$):

Theorem 9 (Parallelity of pairs of vectors in $d > 3$ or $d = 2$)**.** *Let $\Omega := \{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathbb{R}^{dN} : \boldsymbol{x}_i \wedge \boldsymbol{x}_j \neq 0 \ \forall i \neq j\}$. Taking the ground state $f(x) := \prod_{j<k} |B_{jk}|^{-(d-3)}$ one obtains*

$$\int_\Omega |\nabla u|^2 \, dx - (d-3)^2 \int_\Omega \left(\alpha(1-\alpha)\Sigma_1 - \alpha^2 \Sigma_2 \right) |u|^2 \, dx$$
$$= \int_\Omega |\nabla f^{-\alpha} u|^2 f^{2\alpha} \, dx \ \geq \ 0,$$

for all $u \in C_0^\infty(\Omega)$. Hence, if $C_{d,N} := \sup_{x \in \Omega} \Sigma_2(x)/\Sigma_1(x)$, then

$$\int_\Omega |\nabla u|^2 \, dx - \frac{(d-3)^2}{4(1 + C_{d,N})} \int_\Omega \Sigma_1 |u|^2 \, dx \geq \int_\Omega |\nabla f^{-\alpha} u|^2 f^{2\alpha} \, dx \ \geq \ 0,$$

and,

$$\int_\Omega |\nabla u|^2 \, dx - \frac{(d-3)^2}{4C_{d,N}(1 + C_{d,N})} \int_\Omega \Sigma_2 |u|^2 \, dx \geq \int_\Omega |\nabla f^{-\alpha} u|^2 f^{2\alpha} \, dx \ \geq \ 0,$$

with $\alpha := \frac{1}{2(1+C_{d,N})}$ (in both inequalities).

Remark. Note that here $C_{d,N} \leq N-2$ by the Cauchy-Schwarz inequality in $\mathbb{R}^d$. Furthermore, $\operatorname{codim}\Omega^c = dN-(N-2)d-d-1 = d-1$, so the corresponding inequalities hold on $H^1(\mathbb{R}^{dN})$ for $d>2$.

Remark. For $N=2$, we have $\Sigma_2 = 0$ and hence the optimal and sharp 2-particle Hardy inequality

$$\int_{\mathbb{R}^{2d}} |\nabla u|^2\, dx \geq \frac{(d-3)^2}{4} \int_{\mathbb{R}^{2d}} \frac{|\boldsymbol{x}_1|^2+|\boldsymbol{x}_2|^2}{|\boldsymbol{x}_1\wedge\boldsymbol{x}_2|^2}|u|^2\, dx, \tag{20}$$

for $u\in H^1(\mathbb{R}^{2d})$, $d>3$, or $u\in H_0^1(\Omega)$, $d=2$ (cp. Theorem 2).

Proof. One computes (see Appendix A)

$$\nabla_k f = -\frac{(d-3)}{2} f \sum_{j\neq k} |B_{jk}|^{-2}\nabla_k |B_{jk}|^2 = -(d-3) f \sum_{j\neq k} \boldsymbol{x}_j \llcorner (\boldsymbol{x}_j\wedge\boldsymbol{x}_k)^{-1},$$

implying

$$|\nabla f|^2 = (d-3)^2 f^2(\Sigma_1+\Sigma_2),$$

as well as $\Delta f = (d-3)^2 f\Sigma_2$, due to $\Delta_k |B_{kj}|^{-(d-3)} = 0$ on Ω $\forall j$. □

For the critical case $d=3$ we once again need a length scale. One natural way to obtain this in the context of matrix models[1] is to introduce the *matrix model potential* $W := \sum_{j<k}|B_{jk}|^2$ and consider e.g. $\Omega_0 := \Omega\cap\{W<R\}$ for which $H_0^1(\Omega_0) = H_0^1(\{W<R\})$. Defining

$$\tilde{\Sigma}_1(x) := \sum_{j<k} \frac{|\boldsymbol{x}_j|^2+|\boldsymbol{x}_k|^2}{|B_{jk}|^2\big|\ln\frac{1}{R}|B_{jk}|\big|^2},$$

and

$$\tilde{\Sigma}_2(x) := \sum_{i\neq j\neq k\neq i} \frac{\boldsymbol{x}_i \llcorner B_{ik}^{-1}}{\big|\ln\frac{1}{R}|B_{ik}|\big|}\cdot\frac{\boldsymbol{x}_j \llcorner B_{jk}^{-1}}{\big|\ln\frac{1}{R}|B_{jk}|\big|},$$

we then have the following analog of Theorem 9.

Theorem 10 (Parallelity of pairs of vectors in $d=3$)**.** *Taking the ground state* $f(x) := \prod_{j<k}|\ln\frac{1}{R}|B_{jk}||$ *one obtains*

$$\int_{\Omega_0}|\nabla u|^2\,dx - \int_{\Omega_0}\Big(\alpha(1-\alpha)\tilde{\Sigma}_1 - \alpha^2\tilde{\Sigma}_2\Big)|u|^2\,dx$$
$$= \int_{\Omega_0}|\nabla f^{-\alpha}u|^2 f^{2\alpha}\,dx \;\geq\; 0,$$

[1] The case with $d=3$ and N particles corresponds in the matrix models to dimensionally reduced SU(2) Yang-Mills theory from $N+1$ spacetime dimensions.

for all $u \in C_0^\infty(\Omega_0)$. Hence, if $C_{3,N} := \sup_{x\in\Omega_0} \tilde{\Sigma}_2(x)/\tilde{\Sigma}_1(x)$ ($\leq N-2$), then

$$\int_{\Omega_0} |\nabla u|^2\,dx - \frac{1}{4(1+C_{3,N})}\int_{\Omega_0} \tilde{\Sigma}_1|u|^2\,dx \geq \int_{\Omega_0} |\nabla f^{-\alpha}u|^2 f^{2\alpha}\,dx \;\geq\; 0,$$

and,

$$\int_{\Omega_0} |\nabla u|^2\,dx - \frac{1}{4C_{d,N}(1+C_{3,N})}\int_{\Omega_0} \tilde{\Sigma}_2|u|^2\,dx \geq \int_{\Omega_0} |\nabla f^{-\alpha}u|^2 f^{2\alpha}\,dx \;\geq\; 0,$$

with $\alpha := \frac{1}{2(1+C_{3,N})}$ (in both inequalities). In particular, with $N=2$,

$$\int_{\mathbb{R}^6} |\nabla u|^2\,dx \geq \frac{1}{4}\int_{\mathbb{R}^6} \frac{|\boldsymbol{x}_1|^2+|\boldsymbol{x}_2|^2}{|\boldsymbol{x}_1\wedge\boldsymbol{x}_2|^2\left(\ln\frac{1}{R}|\boldsymbol{x}_1\wedge\boldsymbol{x}_2|\right)^2}|u|^2\,dx, \tag{21}$$

for all $u \in H_0^1(\{W<R\})$.

The proof is completely analogous to the proof of Theorem 9, with similar modifications as in Theorem 7.

3.3 Inequalities involving volumes of simplices of points

Consider again a tuple of N vectors in $\mathbb{R}^d$, but now think of them as points. These points span a (possibly degenerate) $N-1$-simplex with volume given by

$$\begin{aligned} V(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N) &:= \frac{1}{(N-1)!}\big|(\boldsymbol{x}_1-\boldsymbol{x}_N)\wedge\ldots\wedge(\boldsymbol{x}_{N-1}-\boldsymbol{x}_N)\big| \\ &= \frac{1}{(N-1)!}\big|\det[(\boldsymbol{x}_j-\boldsymbol{x}_N)\cdot(\boldsymbol{x}_k-\boldsymbol{x}_N)]_{1\leq j,k<N}\big|^{\frac{1}{2}}. \end{aligned}$$

Note that this expression is invariant under any permutation of the points (this is also shown explicitly in Appendix A). We have then the following geometric generalization of the $N=2$ case in Theorem 4 (or 5):

Theorem 11 (Volume of the $N-1$-simplex of N points in $\mathbb{R}^d$, $d>N$ or $d=N-1$)**.** *Consider $\Omega := \{(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)\in\mathbb{R}^{dN} : V(x)>0\}$ and the ground state $f(x) := ((N-1)!\,V(x))^{-(d-N)}$. Denoting*

$$\Sigma^{(N)}(x) := \frac{\sum_{k=1}^N V(\boldsymbol{x}_1,\ldots,\check{\boldsymbol{x}}_k,\ldots,\boldsymbol{x}_N)^2}{(N-1)^2\,V(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)^2} \tag{22}$$

(where ˇ means deletion), we then have

$$\int_\Omega |\nabla u|^2\,dx - \frac{(d-N)^2}{4}\int_\Omega \Sigma^{(N)}|u|^2\,dx = \int_\Omega |\nabla f^{-\frac{1}{2}}u|^2 f\,dx \;\geq\; 0, \tag{23}$$

for all $u\in C_0^\infty(\Omega)$. The corresponding sharp Hardy inequalities hold on $H^1(\mathbb{R}^{dN})$ for $d>N$ and $H_0^1(\Omega)$ for $d=N-1$.

Before proving this theorem, it is convenient to introduce the following notation:

$$\begin{aligned} A_k(x) &:= (-1)^{k-1} \bigwedge_{1\le j\ne k<N} (\boldsymbol{x}_j - \boldsymbol{x}_N), \quad k = 1,\dots,N-1, \\ A_{k=N}(x) &:= (-1)^{N-1} \bigwedge_{1\le j<N-1} (\boldsymbol{x}_j - \boldsymbol{x}_{N-1}), \quad \text{and} \\ A(x) &:= \bigwedge_{1\le j<N} (\boldsymbol{x}_j - \boldsymbol{x}_N), \end{aligned}$$

so that

$$\Sigma^{(N)}(x) = \sum_{k=1}^{N} \left|A_k \llcorner A^{-1}\right|^2 = \sum_{k=1}^{N} |A_k|^2/|A|^2.$$

Note that $\Sigma^{(N)}$ describes a ratio of a mean of squares of volumes of all $N-2$-dimensional subsimplices A_k to the square of the volume of the full simplex A. In particular, for $N = 3$,

$$\Sigma^{(3)} = \frac{r_{12}^2 + r_{23}^2 + r_{31}^2}{4V(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)^2} \ge \frac{4}{27}\frac{\rho^2}{R^4},$$

where we used that the simplex area V is bounded by $\frac{3\sqrt{3}}{4\pi}$ times the area of the circumcircle.

Proof of Theorem 11. Note that we have defined $A_{k<N}$ and A_N s.t. (cp. Appendix A)

$$A = (\boldsymbol{x}_k - \boldsymbol{x}_N) \wedge A_k = (\boldsymbol{x}_N - \boldsymbol{x}_{N-1}) \wedge A_N. \tag{24}$$

For each fixed $k < N$ we then have $f(x) = |(\boldsymbol{x}_k - \boldsymbol{x}_N) \wedge A_k|^{-(d-N)}$ and just as in Theorem 2 that $\nabla_{\boldsymbol{x}_k} f = -(-1)^{\binom{N-1}{2}}(d-N)(A_k \llcorner A^{-1})f$ and $\Delta_{\boldsymbol{x}_k} f = 0$, and hence the following optimal single-particle Hardy GSR:

$$\int_\Omega |\nabla_k u|^2\, dx - \frac{(d-N)^2}{4}\int_\Omega |A_k \llcorner A^{-1}|^2 |u|^2\, dx = \int_\Omega |\nabla_k f^{-\frac{1}{2}} u|^2 f\, dx. \tag{25}$$

For $k = N$ we use the invariance of V under permutations, i.e. (24), and write instead $f(x) = |(\boldsymbol{x}_N - \boldsymbol{x}_{N-1}) \wedge A_N|^{-(d-N)}$, with analogous conclusions. Hence, $|\nabla f|^2 = \sum_k |\nabla_k f|^2 = (d-N)^2 \Sigma^{(N)} f^2$, $\Delta f = \sum_k \Delta_k f = 0$, and the GSR (23) follows. Finally, note that in this case $\operatorname{codim}\Omega^c = dN - (N-1)d - (N-2) = d - N + 2$ and hence $H_0^1(\Omega) = H^1(\mathbb{R}^{dN})$ for $d > N-1$. Sharpness is proven in Appendix B. ☐

Again, for the codimension-critical case $d = N$ we fix a length scale $R > 0$ and restrict to, e.g., $\Omega_R := \{(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N) \in \mathbb{R}^{d^2} : V(x) < R\}$.

Theorem 12 (Volume of the $d-1$-simplex of d points in $\mathbb{R}^d$)**.** *Consider* $\Omega := \{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d) \in \mathbb{R}^{d^2} : 0 < V(x) < R\}$ *and the ground state* $f(x) := \left|\ln \frac{1}{R} V(x)\right|$. *Denoting*

$$\tilde{\Sigma}^{(d)}(x) := \frac{\sum_{k=1}^{d} V(\boldsymbol{x}_1, \ldots, \check{\boldsymbol{x}}_k, \ldots, \boldsymbol{x}_d)^2}{(d-1)^2\, V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d)^2 \left(\ln \frac{1}{R} V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d)\right)^2}, \tag{26}$$

we then have

$$\int_\Omega |\nabla u|^2\, dx - \frac{1}{4} \int_\Omega \tilde{\Sigma}^{(d)} |u|^2\, dx = \int_\Omega |\nabla f^{-\frac{1}{2}} u|^2 f\, dx \ \geq\ 0, \tag{27}$$

for all $u \in C_0^\infty(\Omega)$. *The corresponding Hardy inequality holds on* $H_0^1(\Omega_R)$.

Proof. This is completely analogous to the proof of Theorem 11, using that $\nabla_{\boldsymbol{x}_k} f = -(-1)^{\binom{d-1}{2}} (A_k \llcorner A^{-1})$ and $\Delta_{\boldsymbol{x}_k} f = 0$. □

Now, whereas Theorem 9 involved all possible quantities $|B_{ij}| = |(\boldsymbol{x}_i - 0) \wedge (\boldsymbol{x}_j - 0)|$ among N points, i.e. (twice) the volumes of all 2-simplices spanned by selections of points of the form $\{\boldsymbol{x}_i, \boldsymbol{x}_j, 0\}$, this can naturally be generalized to higher dimensions as follows. This is both a geometrically and combinatorially more complete generalization of Theorem 5, which corresponds to $p = 2$:

Theorem 13 (Volumes of all simplices of p points among N points in $d \neq p$)**.** *Consider* $\Omega := \{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathbb{R}^{dN} : \mathbb{V}(x) > 0\}$ *where*

$$\mathbb{V}(x) := \frac{1}{((p-1)!)^{\binom{N}{p}}} \prod_{1 \leq j_1 < \ldots < j_p \leq N} \left|(\boldsymbol{x}_{j_1} - \boldsymbol{x}_{j_p}) \wedge \ldots \wedge (\boldsymbol{x}_{j_{p-1}} - \boldsymbol{x}_{j_p})\right|$$

is the product of the volumes of all $p-1$-simplices in $\mathbb{R}^d$ spanned by p of the points $\{\boldsymbol{x}_{j=1,\ldots,N}\}$, *and take the ground state* $f := \left(((p-1)!)^{\binom{N}{p}} \mathbb{V}\right)^{-(d-p)}$.

Denote by $\Lambda = \Lambda(p, N)$ *the set of ordered subsets* $\lambda = (\lambda_1, \ldots, \lambda_p) \subseteq \{1, \ldots, N\}$ *of p elements out of N. For $\lambda \in \Lambda$ define the $p-1$-blade* $A_\lambda := A(\boldsymbol{x}_{\lambda_1}, \ldots, \boldsymbol{x}_{\lambda_p})$, *and for each $k \in \lambda$ let $A_{\lambda,k}$ denote a $p-2$-blade s.t.* $A_\lambda = (\boldsymbol{x}_k - \boldsymbol{x}_{\lambda_q}) \wedge A_{\lambda,k}$ *for some* $\lambda_q \in \lambda \setminus k$. *With*

$$\Sigma_1^{(p,N)}(x) := \sum_{k=1}^{N} \sum_{\lambda \in \Lambda \,:\, k \in \lambda} \left|A_{\lambda,k} \llcorner A_\lambda^{-1}\right|^2 = \sum_{\lambda \in \Lambda} \sum_{k \in \lambda} |A_{\lambda,k}|^2 / |A_\lambda|^2,$$

and

$$\Sigma_2^{(p,N)}(x) := \sum_{k=1}^{N} \sum_{\substack{\lambda,\mu \in \Lambda : \\ k \in \lambda \neq \mu \ni k}} \left(A_{\lambda,k} \llcorner A_\lambda^{-1}\right) \cdot \left(A_{\mu,k} \llcorner A_\mu^{-1}\right),$$

we then have

$$\int_\Omega |\nabla u|^2\,dx - (d-p)^2 \int_\Omega \Big(\alpha(1-\alpha)\Sigma_1^{(p,N)} - \alpha^2\Sigma_2^{(p,N)}\Big)|u|^2\,dx$$
$$= \int_\Omega |\nabla f^{-\alpha}u|^2 f^{2\alpha}\,dx \;\geq\; 0, \qquad (28)$$

for all $u \in C_0^\infty(\Omega)$. *Hence, if* $C_{d,N}^{(p)} := \sup_{x\in\Omega} \Sigma_2^{(p,N)}(x)/\Sigma_1^{(p,N)}(x)$, *then*

$$\int_\Omega |\nabla u|^2\,dx - \frac{(d-p)^2}{4(1+C_{d,N}^{(p)})}\int_\Omega \Sigma_1^{(p,N)}|u|^2\,dx \geq \int_\Omega |\nabla f^{-\alpha}u|^2 f^{2\alpha}\,dx \;\geq\; 0, \qquad (29)$$

and,

$$\int_\Omega |\nabla u|^2\,dx - \frac{(d-p)^2}{4C_{d,N}^{(p)}(1+C_{d,N}^{(p)})}\int_\Omega \Sigma_2^{(p,N)}|u|^2\,dx \geq \int_\Omega |\nabla f^{-\alpha}u|^2 f^{2\alpha}\,dx \;\geq\; 0, \qquad (30)$$

with $\alpha := \frac{1}{2(1+C_{d,N}^{(p)})}$.

Remark. Note that by Cauchy-Schwarz in $\mathbb{R}^d$,

$$\begin{aligned}
\Sigma_2^{(p,N)} &\leq \sum_k \sum_{\lambda\ni k} \sum_{\substack{\mu\ni k\\ \mu\neq\lambda}} |A_{\lambda,k} \llcorner A_\lambda^{-1}||A_{\mu,k} \llcorner A_\mu^{-1}| \\
&\leq \sum_k \sum_{\lambda\ni k} \sum_{\substack{\mu\ni k\\ \mu\neq\lambda}} \frac{1}{2}\big(|A_{\lambda,k} \llcorner A_\lambda^{-1}|^2 + |A_{\mu,k} \llcorner A_\mu^{-1}|^2\big) \\
&= \sum_k \sum_{\lambda\ni k} |A_{\lambda,k} \llcorner A_\lambda^{-1}|^2 \cdot \sum_{\substack{\mu\ni k\\ \mu\neq\lambda}} 1 \;=\; \left(\binom{N-1}{p-1} - 1\right)\Sigma_1^{(p,N)},
\end{aligned}$$

hence $C_{d,N}^{(p)} \leq \binom{N-1}{p-1} - 1$. Furthermore, as in the single-volume case one finds that $\operatorname{codim}\Omega^c = d-p+2$.

Corollary. *For* $d > p$ *we have the generalized many-particle Hardy inequality*

$$\int_{\mathbb{R}^{dN}} |\nabla u|^2\,dx - \frac{(d-p)^2}{4(1+C_{d,N}^{(p)})}\int_{\mathbb{R}^{dN}} \sum_{\lambda\in\Lambda(p,N)} \frac{\sum_{k\in\lambda} V(\boldsymbol{x}_{\lambda_1},\ldots,\check{\boldsymbol{x}}_k,\ldots,\boldsymbol{x}_{\lambda_p})^2}{(p-1)^2\, V(\boldsymbol{x}_{\lambda_1},\ldots,\boldsymbol{x}_{\lambda_p})^2}|u|^2\,dx$$
$$\geq\; 0,$$

for all $u \in H^1(\mathbb{R}^{dN})$. *The inequality holds for* $u \in H_0^1(\Omega)$ *when* $d = p-1$.

Proof. We have $f = \prod_{\lambda \in \Lambda(p,N)} |A_\lambda|^{-(d-p)}$ with $\Delta_k |A_\lambda|^{-(d-p)} = 0$ on $\Omega\ \forall k, \lambda$, and hence

$$\begin{aligned}
\nabla_k f &= \sum_{\lambda \in \Lambda(p,N)\,:\,k \in \lambda} \nabla_k |A_\lambda|^{-(d-p)} \prod_{\mu \in \Lambda(p,N)\,:\,\mu \neq \lambda} |A_\mu|^{-(d-p)} \\
&= -\frac{(d-p)}{2} f \sum_{\lambda \in \Lambda(p,N)\,:\,k \in \lambda} |A_\lambda|^{-2} \nabla_k |A_\lambda|^2,
\end{aligned}$$

and

$$\begin{aligned}
\Delta_k f &= \sum_{\lambda \ni k} \nabla_k |A_\lambda|^{-(d-p)} \cdot \sum_{\substack{\mu \ni k\,: \\ \mu \neq \lambda}} \nabla_k |A_\mu|^{-(d-p)} \prod_{\nu \neq \lambda, \mu} |A_\nu|^{-(d-p)} \\
&= \frac{(d-p)^2}{4} f \sum_{\lambda \ni k} \sum_{\substack{\mu \ni k\,: \\ \mu \neq \lambda}} |A_\lambda|^{-2} \nabla_k |A_\lambda|^2 \cdot |A_\mu|^{-2} \nabla_k |A_\mu|^2.
\end{aligned}$$

Then, again using (31) from Appendix A,

$$\nabla_k |A_\lambda|^2 = \nabla_k |(\boldsymbol{x}_k - \boldsymbol{x}_{\lambda_q}) \wedge A_{\lambda,k}|^2 = (-1)^{\binom{p-1}{2}} 2 A_{\lambda,k} \llcorner A_\lambda,$$

we therefore obtain

$$|\nabla f|^2 = \sum_k |\nabla_k f|^2 = (d-p)^2 f^2 \left(\Sigma_1^{(p,N)} + \Sigma_2^{(p,N)} \right)$$

and

$$\Delta f = \sum_k \Delta_k f = (d-p)^2 f \Sigma_2^{(p,N)}.$$

The GSR (28) and inequalities (29)-(30) now follow as in the earlier theorems. □

Remark. For the optimal large-N dependence of the constants in these many-particle inequalities, it becomes relevant to study the ratio of the geometric quantities w.r.t. the optimal asymptotic probability distribution ρ of points in $\mathbb{R}^d$;

$$C_d^{(p,q)} := \sup_{\rho \geq 0\,:\,\int_{\mathbb{R}^d} d\rho(\boldsymbol{x}) = 1} \frac{\int_{\mathbb{R}^{qd}} \Xi^{(p,q)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_q) \prod_{k=1}^q d\rho(\boldsymbol{x}_k)}{\int_{\mathbb{R}^{pd}} \Sigma^{(p)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p) \prod_{k=1}^p d\rho(\boldsymbol{x}_k)},$$

where $q = p+1, \ldots, 2p-1$, $\Sigma^{(p)}$ was defined in (22), and

$$\Xi^{(p,q)} := \sum_{\substack{1 \in \lambda, \mu \in \Lambda(p,q) \\ : |\lambda \cap \mu| = 2p-q}} \sum_{\pi \in S_q} \left(A_{\pi(\lambda),\pi(1)} \llcorner A_{\pi(\lambda)}^{-1} \right) \cdot \left(A_{\pi(\mu),\pi(1)} \llcorner A_{\pi(\mu)}^{-1} \right)$$

are higher-dimensional generalizations of a single circumradius contribution R_{123}^{-2}. Related optimizations involving lower-dimensional geometric quantities have been discussed in Remarks 2.2(iv) in [3] (see also [12]), 3.1 in [2], and also e.g. (2) in [11].

It is of course possible to generalize Theorem 13 even further and consider the volumes of *all* simplices among N points (i.e. all simplex dimensions simultaneously), including the case of critical codimension. We will not state the corresponding theorem explicitly here.

Appendix A: Some applications of geometric algebra

The *geometric algebra* over $\mathbb{R}^d$ is the exterior algebra $\bigwedge \mathbb{R}^d$ together with the left- and right-interior products $(A, B) \mapsto A \llcorner B$, $(A, B) \mapsto A \lrcorner B$ and the associative Clifford product $(A, B) \mapsto AB$, which are inherited from the usual Euclidean scalar product $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x} \cdot \boldsymbol{y}$. A *p-blade* A is an exterior product of p vectors and is uniquely determined by its corresponding p-dimensional subspace $\bar{A}$, orientation, and magnitude $|A| := \sqrt{AA^\dagger}$, where $A^\dagger := (-1)^{\binom{p}{2}} A$ is the *reverse* of A. Note that if A, B are blades then their exterior and interior products $A \wedge B$ and $A \llcorner B$ are blades as well, and that $A \llcorner B = AB$ if $\bar{A} \subseteq \bar{B}$ (see e.g. Section 3 in [9]).

If A is a p-blade then the gradient

$$\nabla_{\boldsymbol{x}} |\boldsymbol{x} \wedge A|^2 = 2(\boldsymbol{x} \wedge A)A^\dagger = 2A^\dagger(A \wedge \boldsymbol{x}) = (-1)^{\binom{p+1}{2}} 2A \llcorner (\boldsymbol{x} \wedge A). \tag{31}$$

One way to see this is to note that it is trivially true for $A = 0$, and that for $A \neq 0$ we can write any point $\boldsymbol{x}$ in $\mathbb{R}^d$ uniquely as

$$\boldsymbol{x} = \boldsymbol{x}AA^{-1} = (\boldsymbol{x} \llcorner A)A^{-1} + (\boldsymbol{x} \wedge A)A^{-1} = \boldsymbol{x}_\parallel + \boldsymbol{x}_\perp,$$

where $\boldsymbol{x}_\parallel := P_{\bar{A}}\boldsymbol{x}$ is the orthogonal projection on $\bar{A}$ and $\boldsymbol{x}_\perp := (1 - P_{\bar{A}})\boldsymbol{x}$ (the so-called *rejection* on $\bar{A}$; see e.g. Section 3.3 in [9]). Hence,

$$\begin{aligned}
\nabla_{\boldsymbol{x}} |\boldsymbol{x} \wedge A|^2 &= \nabla_{\boldsymbol{x}} |(\boldsymbol{x} \wedge A)A^{-1}|^2 |A|^2 \\
&= \nabla_{\boldsymbol{x}_\parallel} |\boldsymbol{x}_\perp|^2 |A|^2 + \nabla_{\boldsymbol{x}_\perp} |\boldsymbol{x}_\perp|^2 |A|^2 \\
&= 0 + 2\boldsymbol{x}_\perp |A|^2 \;=\; 2(\boldsymbol{x} \wedge A)A^\dagger.
\end{aligned}$$

The other equalities in (31) follow by taking the reverse.

It then also follows that

$$\left|\nabla_{\boldsymbol{x}} |\boldsymbol{x} \wedge A|^2\right|^2 = 4\left((\boldsymbol{x} \wedge A)A^\dagger\right)\left((\boldsymbol{x} \wedge A)A^\dagger\right)^\dagger = 4|A|^2 |\boldsymbol{x} \wedge A|^2,$$

$$\begin{aligned}
\nabla_{\boldsymbol{x}} |\boldsymbol{x} \wedge A|^2 \cdot \nabla_{\boldsymbol{y}} |\boldsymbol{y} \wedge B|^2 &= 4((\boldsymbol{x} \wedge A)A^\dagger) \cdot ((\boldsymbol{y} \wedge B)B^\dagger) \\
&= 4(A^\dagger \llcorner (A \wedge \boldsymbol{x})) \cdot (B^\dagger \llcorner (B \wedge \boldsymbol{y})) \\
&= (-1)^{\binom{p+1}{2} + \binom{q+1}{2}} 4(A \llcorner (\boldsymbol{x} \wedge A)) \cdot (B \llcorner (\boldsymbol{y} \wedge B)),
\end{aligned}$$

for A, B p- resp. q-blades, and, by e.g. Exercise 3.6 in [9],

$$\Delta_{\boldsymbol{x}} |\boldsymbol{x} \wedge A|^2 = 2\nabla_{\boldsymbol{x}} (\boldsymbol{x} \wedge A) A^\dagger = 2 \sum_{k=1}^{d} e_k (e_k \wedge A) A^\dagger = 2(d-p)|A|^2,$$

as well as (outside the support of the corresponding distribution)

$$\begin{aligned} \Delta_{\boldsymbol{x}} |\boldsymbol{x} \wedge A|^{2\beta} &= \beta(\beta-1)|\boldsymbol{x} \wedge A|^{2\beta-4} \left|\nabla_{\boldsymbol{x}} |\boldsymbol{x} \wedge A|^2\right|^2 \\ &\quad + \beta |\boldsymbol{x} \wedge A|^{2\beta-2} \Delta_{\boldsymbol{x}} |\boldsymbol{x} \wedge A|^2 \\ &= 2\beta(2\beta + d - p - 2)|A|^2 |\boldsymbol{x} \wedge A|^{2\beta-2}, \end{aligned}$$

which is zero for $\beta = -(d-p-2)/2$. Furthermore, (again on a suitable domain)

$$\nabla_{\boldsymbol{x}} \ln \frac{1}{R} |\boldsymbol{x} \wedge A| = \frac{1}{2} |\boldsymbol{x} \wedge A|^{-2} \nabla_{\boldsymbol{x}} |\boldsymbol{x} \wedge A|^2 = (\boldsymbol{x} \wedge A)^{-1} A^\dagger,$$

$$\Delta_{\boldsymbol{x}} \ln \frac{1}{R} |\boldsymbol{x} \wedge A| = \frac{1}{2} \nabla_{\boldsymbol{x}} \cdot \left(|\boldsymbol{x} \wedge A|^{-2} \nabla_{\boldsymbol{x}} |\boldsymbol{x} \wedge A|^2 \right) = (d-p-2)|A|^2 |\boldsymbol{x} \wedge A|^{-2}.$$

Lastly, we have the following explicit invariance of the simplex volume under permutations of the points:

Proposition 14. *Let $A := \bigwedge_{1 \le j < N} (\boldsymbol{x}_j - \boldsymbol{x}_N)$ be the $N-1$-blade associated to the points $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$. Then A is invariant, up to a sign, under any permutation of the points.*

Proof. A is due to the total antisymmetry of the exterior product clearly invariant under any permutation σ of $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1})$, up to a sign ($\operatorname{sgn} \sigma$). Furthermore, for any $1 \le k < N$,

$$\begin{aligned} A &= (\boldsymbol{x}_1 - \boldsymbol{x}_N) \wedge \ldots \wedge (\boldsymbol{x}_k - \boldsymbol{x}_N) \wedge \ldots \wedge (\boldsymbol{x}_{N-1} - \boldsymbol{x}_N) = \\ &= \bigwedge_{j=1}^{k-1} (\boldsymbol{x}_j - \boldsymbol{x}_k + \boldsymbol{x}_k - \boldsymbol{x}_N) \wedge (\boldsymbol{x}_k - \boldsymbol{x}_N) \wedge \bigwedge_{j=k+1}^{N-1} (\boldsymbol{x}_j - \boldsymbol{x}_k + \boldsymbol{x}_k - \boldsymbol{x}_N) \\ &= \bigwedge_{j=1}^{k-1} (\boldsymbol{x}_j - \boldsymbol{x}_k) \wedge (\boldsymbol{x}_k - \boldsymbol{x}_N) \wedge \bigwedge_{j=k+1}^{N-1} (\boldsymbol{x}_j - \boldsymbol{x}_k) \\ &= (-1)^{N-k} \bigwedge_{\substack{j=1 \\ j \ne k}}^{N} (\boldsymbol{x}_j - \boldsymbol{x}_k), \end{aligned}$$

by multilinearity and antisymmetry. The proposition then follows by composition of permutations. □

Appendix B: Sharpness of derived constants

For completeness, we briefly note in this appendix how the explicit ground states f in the derived Hardy GSRs also can be used to determine sharpness of the constants in the corresponding Hardy inequalities.

Lemma 15. *Suppose that Ω and f in Proposition 1 are such that $\overline{\Omega} = \mathbb{R}^n$ and*

i) $\int_{\mathbb{R}^n} f^{1-\delta} e^{-|x|}\, dx$ *is uniformly bounded for small $\delta > 0$,*

ii) $\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f^2} f^{1-\delta} e^{-|x|}\, dx$ *is finite for small $\delta > 0$, but $\to \infty$ as $\delta \to 0$.*

Then, for every $\epsilon > 0$ there exists $u_\delta := f^{\frac{1}{2}} f^{-\frac{\delta}{2}} e^{-\frac{1}{2}|x|} \in H^1(\mathbb{R}^n)$, with $\delta > 0$, s.t.

$$\int_{\mathbb{R}^n} |\nabla u_\delta|^2\, dx \leq \left(\frac{1}{4} + \epsilon\right) \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f^2} |u_\delta|^2\, dx. \tag{32}$$

Remark. If $f(x) \to 0$ as $x \to \partial\Omega$, then we reverse the sign on δ above and note that $u_\delta \in H_0^1(\Omega)$ for small δ.

Proof. Using that $u_\delta \in C^1(\Omega \setminus \{0\})$ and

$$|\nabla u_\delta| = \left| \frac{\nabla f}{2f} u_\delta - \frac{\delta \nabla f}{2f} u_\delta - \frac{\nabla |x|}{2} u_\delta \right| \leq \frac{1 + |\delta|}{2} \frac{|\nabla f|}{f} |u_\delta| + \frac{1}{2} |u_\delta|$$

on $\Omega \setminus \{0\}$, we find by *i)* and *ii)* that $u_\delta \in H^1(\mathbb{R}^n)$ for sufficiently small $(\overset{+}{-})\delta > 0$, and that

$$\begin{aligned} \|\nabla u_\delta\|_{L^2} &\leq \left(\frac{1}{2} + \frac{|\delta|}{2}\right) \left\| \frac{\nabla f}{f} u_\delta \right\|_{L^2} + \frac{1}{2} \|u_\delta\|_{L^2} \\ &\leq \left(\frac{1}{2} + \frac{|\delta|}{2} + \frac{\sqrt{C}}{2} \left\| \frac{\nabla f}{f} u_\delta \right\|_{L^2}^{-1} \right) \left\| \frac{\nabla f}{f} u_\delta \right\|_{L^2}, \end{aligned}$$

where $C < \infty$ denotes a bound for *i)*. The result follows by taking $\delta \to 0$. $\square$

The conditions *i)* and *ii)* above typically hold when f is of the form $f \sim \delta_\Omega^{-(k-2)}$ near $\partial\Omega$, where δ_Ω is the distance to Ω^c and k the codimension of Ω^c. Let us prove this explicitly for some of the Hardy GSRs considered in this paper.

Proposition 16. *The constant $(d-p-2)^2/4$ in (6) is sharp.*

Proof. As in Appendix A, we split the space $\mathbb{R}^d$ into variables $\boldsymbol{x}_\parallel \in \bar{A}$, and $\boldsymbol{x}_\perp$ orthogonal to $\bar{A}$. Then, since $f(\boldsymbol{x}) \propto |\boldsymbol{x}_\perp|^{-(d-p-2)}$,

$$\begin{aligned} \int_{\mathbb{R}^d} f^{1-\delta} e^{-|\boldsymbol{x}|} dx &\propto \int_{\mathbb{R}^p} \int_{\mathbb{R}^{d-p}} |\boldsymbol{x}_\perp|^{-(d-p-2)(1-\delta)} e^{-|\boldsymbol{x}|}\, d\boldsymbol{x}_\perp\, d\boldsymbol{x}_\parallel \\ &\leq |S^{d-p-1}| \int_{\mathbb{R}^p} e^{-\frac{1}{2}|\boldsymbol{x}_\parallel|}\, d\boldsymbol{x}_\parallel \int_{r=0}^{\infty} r^{-(d-p-2)(1-\delta)} e^{-\frac{r}{2}} r^{d-p-1}\, dr, \end{aligned}$$

which is uniformly bounded for $0 < |\delta| < \delta_0$, and

$$\int_{\mathbb{R}^d} \frac{|\nabla f|^2}{f^2} f^{1-\delta} e^{-|\boldsymbol{x}|} dx \propto \int_{\mathbb{R}^p} \int_{\mathbb{R}^{d-p}} |\boldsymbol{x}_\perp|^{-2-(d-p-2)(1-\delta)} e^{-|\boldsymbol{x}|}\, d\boldsymbol{x}_\perp\, d\boldsymbol{x}_\parallel$$
$$\propto \int_{r=0}^{\infty} \int_{\mathbb{R}^p} e^{-\sqrt{|\boldsymbol{x}_\parallel|^2 + r^2}}\, d\boldsymbol{x}_\parallel\, r^{-1+(d-p-2)\delta}\, dr,$$

which is finite for $0 < \binom{+}{-}\delta < \delta_0$, but tends to infinity when $\delta \to 0$. Hence, by Lemma 15, the constant in the GSR potential $\frac{|\nabla f|^2}{4f^2} = \frac{(d-p-2)^2}{4} \frac{1}{|\boldsymbol{x}_\perp|^2}$ is sharp for $d \neq p + 2$. $\square$

Proposition 17. *The constant* $N((N-1)d-2)^2/4$ *in* (8) *is sharp for* $(N-1)d > 2$.

Proof. The set Ω^c in Theorem 4 is a linear subspace of $\mathbb{R}^{dN}$ which can be parameterized by, say, $\boldsymbol{x}_N \in \mathbb{R}^d$. We then have the ground state

$$f(x) := \rho^{4\alpha} = \left(\sum_{i<N} |\boldsymbol{y}_i|^2 + \sum_{i<j<N} |\boldsymbol{y}_i - \boldsymbol{y}_j|^2 \right)^{-\frac{(N-1)d-2}{2}},$$

where we for fixed $\boldsymbol{x}_N$ define $\boldsymbol{y}_i := \boldsymbol{x}_i - \boldsymbol{x}_N$. Hence, using that

$$|y|^2 := \sum_{i<N} |\boldsymbol{y}_i|^2 \;\le\; \rho^2 \;\le\; \sum_{i<N} |\boldsymbol{y}_i|^2 + \sum_{i<j<N} \left(|\boldsymbol{y}_i|^2 + |\boldsymbol{y}_j|^2 \right) \le C|y|^2$$

(here and in the following, C will denote some unspecified positive constants), we find

$$\int_{\mathbb{R}^{dN}} f^{1-\delta} e^{-|x|} dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^{(N-1)d}} \rho^{-((N-1)d-2)(1-\delta)} e^{-|x|}\, dy\, d\boldsymbol{x}_N$$
$$\le C \int_{\mathbb{R}^d} \int_{\mathbb{R}^{(N-1)d}} e^{-\sqrt{|x'|^2 + |\boldsymbol{x}_N|^2}} |y|^{-((N-1)d-2)(1-\delta)}\, dy\, d\boldsymbol{x}_N$$

(where $x = (x', \boldsymbol{x}_N)$), which is uniformly bounded for $0 < \delta < \delta_0$, and

$$\int_{\mathbb{R}^{dN}} \frac{|\nabla f|^2}{f^2} f^{1-\delta} e^{-|x|} dx$$
$$\sim C \int_{\mathbb{R}^d} \int_{\mathbb{R}^{(N-1)d}} e^{-\sqrt{|x'|^2 + |\boldsymbol{x}_N|^2}} |y|^{-2-((N-1)d-2)(1-\delta)}\, dy\, d\boldsymbol{x}_N,$$

which is finite for $0 < \delta < \delta_0$, but tends to infinity as $\delta \to 0$. Hence, the GSR constant in (8) is sharp by Lemma 15. $\square$

Proposition 18. *The constant* $(d-3)^2/4$ *in* (20) *is sharp.*

Proof. Here $f(x) = |\boldsymbol{x}_1 \wedge \boldsymbol{x}_2|^{-(d-3)}$ and $\Omega^c = \{(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathbb{R}^{2d} : \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 = 0\}$ is a cone-like set which can be parameterized by $\boldsymbol{x}_1 \in \mathbb{R}^d \setminus \{0\}$, $\boldsymbol{x}_2 \in \mathbb{R}\, \boldsymbol{x}_1$, and $\boldsymbol{x}_1 = 0$, $\boldsymbol{x}_2 \in \mathbb{R}^d$. Hence, for each fixed $\boldsymbol{x}_1 \neq 0$ we split the second variable into $x_{2\|}$ along, and $\boldsymbol{x}_{2\perp}$ orthogonal to, the line $\mathbb{R}\, \boldsymbol{x}_1$, write $|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2| = |\boldsymbol{x}_1||\boldsymbol{x}_{2\perp}|$, and deduce (with $\delta' := (d-3)\delta$)

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} f^{1-\delta} e^{-|x|} dx \\ &\qquad = \int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} e^{-|x|} dx_{2\|}\, |\boldsymbol{x}_{2\perp}|^{-(d-3)+\delta'} d\boldsymbol{x}_{2\perp}\, |\boldsymbol{x}_1|^{-(d-3)+\delta'}\, d\boldsymbol{x}_1,\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} \frac{|\nabla f|^2}{f^2} f^{1-\delta} e^{-|x|} dx \\ &\propto \int_{\mathbb{R}^{2d}} \frac{|\boldsymbol{x}_1|^2}{|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2|^2} f^{1-\delta} e^{-|x|} dx + \int_{\mathbb{R}^{2d}} \frac{|\boldsymbol{x}_2|^2}{|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2|^2} f^{1-\delta} e^{-|x|} dx \\ &= 2 \int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} e^{-|x|} dx_{2\|}\, |\boldsymbol{x}_{2\perp}|^{-2-(d-3)+\delta'} d\boldsymbol{x}_{2\perp}\, |\boldsymbol{x}_1|^{-(d-3)+\delta'}\, d\boldsymbol{x}_1,\end{aligned}$$

with similar conclusions as in our earlier examples. □

Proposition 19. *The constant* $(d-N)^2/4$ *in* (23) *is sharp.*

Proof. Here $f(x) = |A|^{-(d-N)}$, with $A = \bigwedge_{j=1}^{N-1} (\boldsymbol{x}_j - \boldsymbol{x}_N)$, and $\Omega^c = \{x \in \mathbb{R}^{dN} : A = 0\}$ can be parameterized by $\boldsymbol{x}_N \in \mathbb{R}^d$ and, defining $\boldsymbol{y}_j := \boldsymbol{x}_j - \boldsymbol{x}_N$ for each fixed $\boldsymbol{x}_N$, by

$$\boldsymbol{y}_1 \in \mathbb{R}^d \setminus \{0\}, \quad \boldsymbol{y}_2 \in \mathbb{R}^d \setminus \mathbb{R}\, \boldsymbol{y}_1, \quad \boldsymbol{y}_3 \in \mathbb{R}^d \setminus \overline{\boldsymbol{y}_1 \wedge \boldsymbol{y}_2}, \quad \ldots, \quad \boldsymbol{y}_{N-1} \in \overline{B}, \tag{33}$$

with $B := \boldsymbol{y}_1 \wedge \ldots \wedge \boldsymbol{y}_{N-2}$, *plus,* $\boldsymbol{y}_1 = 0$, $\boldsymbol{y}_{j=2,\ldots,N-1} \in \mathbb{R}^d$, and so forth.

Hence, for each fixed $\boldsymbol{x}_N \in \mathbb{R}^d$ and $\boldsymbol{y}_{j=1,\ldots,N-2}$ in general position as in (33) we split the last variable $\boldsymbol{y}_{N-1}$ into $\boldsymbol{y}_\| \in \overline{B}$ and $\boldsymbol{y}_\perp$ orthogonal to $\overline{B}$, write $|A| = |B \wedge \boldsymbol{y}_{N-1}| = |B||\boldsymbol{y}_\perp|$, and deduce (with $\delta' := (d-N)\delta$)

$$\begin{aligned}\int_{\mathbb{R}^{dN}} f^{1-\delta} e^{-|x|} dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} \cdots \int_{\mathbb{R}^d \setminus \overline{\boldsymbol{y}_1 \wedge \ldots \wedge \boldsymbol{y}_{N-3}}} \int_{\mathbb{R}^{d-N+2}} \int_{\mathbb{R}^{N-2}} e^{-|x|} d\boldsymbol{y}_\| \\ \cdot\, |\boldsymbol{y}_\perp|^{-(d-N)+\delta'} d\boldsymbol{y}_\perp\, |B|^{-(d-N)+\delta'}\, d\boldsymbol{y}_{N-2} \ldots d\boldsymbol{y}_1\, d\boldsymbol{x}_N,\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^{dN}} \frac{|\nabla f|^2}{f^2} f^{1-\delta} e^{-|x|} dx \;\propto\; \int_{\mathbb{R}^{dN}} \Sigma^{(N)} f^{1-\delta} e^{-|x|} dx \\ = N \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} \cdots \int_{\mathbb{R}^d \setminus \overline{\boldsymbol{y}_1 \wedge \ldots \wedge \boldsymbol{y}_{N-3}}} \int_{\mathbb{R}^{d-N+2}} \int_{\mathbb{R}^{N-2}} e^{-|x|} d\boldsymbol{y}_\| \\ \cdot\; |\boldsymbol{y}_\perp|^{-2-(d-N)+\delta'} d\boldsymbol{y}_\perp\, |B|^{-(d-N)+\delta'}\, d\boldsymbol{y}_{N-2} \ldots d\boldsymbol{y}_1\, d\boldsymbol{x}_N,\end{aligned}$$

with similar conclusions as in our earlier examples. □

References

[1] E. B. Davies, *A review of Hardy inequalities*, The Mazya anniversary collection, vol. 2, Operator Theory: Advances and Applications 110 (Birkhäuser, Basel, 1999) 55–67.

[2] R. L. Frank, T. Hoffman-Ostenhof, A. Laptev, J. P. Solovej, *Hardy inequalities for large fermionic systems*, 2006, unpublished.

[3] M. Hoffmann-Ostenhof, T. Hoffman-Ostenhof, A. Laptev, J. Tidblom, *Many-particle Hardy Inequalities*, J. London Math. Soc. (2) 77 (2008) 99-114.

[4] A. Kufner, B. Opic, *Hardy-type inequalities*, Pitman Research Notes in Mathematics 219 (Longman Scientific & Technical, Harlow, 1990).

[5] A. Laptev, personal communication.

[6] E. Lieb, R. Seiringer, *The stability of matter in quantum mechanics*, (Cambridge University Press, Cambridge, 2010).

[7] D. Lundholm, *Zero-energy states in supersymmetric matrix models*, Ph.D. thesis, KTH, 2010, `http://urn.kb.se/resolve?urn=urn:nbn:se:kth:diva-12846`

[8] D. Lundholm, F. Portmann, in preparation.

[9] D. Lundholm, L. Svensson, *Clifford algebra, geometric algebra, and applications*, `arXiv:0907.5356`.

[10] V. G. Maz'ya, *Sobolev spaces*, (Springer, Berlin, 1985).

[11] P. T. Nam, *New bounds on the maximum ionization of atoms*, `arXiv:1009.2367`.

[12] J. Tidblom, *Improved L^p Hardy Inequalities*, Ph.D. thesis, Stockholm University, 2005 `http://urn.kb.se/resolve?urn=urn:nbn:se:su:diva-615`